The scattering matrix

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$$[\phi(t,\bar{x}), \pi(t,y)] = i\,\delta^{(3)}(\bar{x}-\bar{y}), \qquad [\phi(t,\bar{x}), \phi(t,\bar{y})] = [\pi(t,\bar{x}), \pi(t,\bar{y})] = 0$$

Let us derive an equation of motion for the quantum fields $\phi(t,\bar{x})$ and $\pi(t,\bar{x})$

$$\dot{\phi}(t,\bar{x}) = -i\,[\phi(t,\bar{x}), H]$$

$$H = \int d^3y \left\{ \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 \right\}, \qquad \dot{\phi} = \pi$$

$$\left[\phi(t,\bar{x}), \int d^3y\, \frac{1}{2}\dot{\phi}(t,y)^2\right] = \frac{1}{2}\int d^3y\, [\phi(t,\bar{x}), \pi^2(t,\bar{y})] =$$

$$[a,b^2] = ab^2 - b^2a = (ab-ba)b - b(ba-ab) =$$

$$= [a,b]b - b[b,a] =$$

$$= [a,b]b + b[a,b]$$

$$= \frac{1}{2}\int d^3y \left\{ [\phi(t,\bar{x}), \pi(t,\bar{y})]\,\pi(t,\bar{y}) + \pi(t,\bar{y})[\phi(t,\bar{x}), \pi(t,\bar{y})] \right\} =$$

$$= \frac{i}{2}\int d^3y \left\{ \delta^{(3)}(\bar{x}-\bar{y})\,\pi(t,\bar{y}) + \pi(t,\bar{y})\,\delta^{(3)}(\bar{x}-\bar{y}) \right\} = i\,\pi(t,\bar{x})$$

$$\left[\phi(t,\bar{x}), \int d^3y\, \frac{1}{2}(\bar{\nabla}_y\phi)^2\right] = \frac{1}{2}\int d^3y\, [\phi(t,\bar{x}), (\bar{\nabla}_y\phi)^2] = \frac{1}{2}\int d^3y \left\{ [\phi(t,\bar{x}), \bar{\nabla}_y\phi(t,\bar{y})]\cdot\bar{\nabla}_y\phi(t,y) + \right.$$

$$\left. + \bar{\nabla}_y\phi(t,\bar{y})\,[\phi(t,\bar{x}), \bar{\nabla}_y\phi(t,\bar{y})] \right\} =$$

$$= \frac{1}{2}\int d^3y \left\{ \bar{\nabla}_y\Big(\underbrace{[\phi(t,\bar{x}), \phi(t,\bar{y})]}_{0}\Big)\cdot\bar{\nabla}\phi(t,\bar{y}) + \bar{\nabla}_y\phi(t,\bar{y})\cdot\bar{\nabla}_y\Big(\underbrace{[\phi(t,\bar{x}), \phi(t,\bar{y})]}_{0}\Big) \right\} = 0$$

also $\left[\phi(t,\bar{x}), \int d^3y\, \frac{1}{2}\phi(t,y)^2\right] = 0$ so also $\left[\phi(t,\bar{x}), \int d^3y\, \frac{1}{2}\phi(t,y)^2\right] = 0$, so

$$\Pi(t,\bar{x}) = \boxed{\dot{\phi}(t,\bar{x}) = -i\left[\phi(t,\bar{x}), H\right]}$$

$$\left[\Pi(t,\bar{x}), H\right] = \frac{1}{2}\int d^3y \left[\Pi(t,\bar{x}), \left(\Pi(t,\bar{y})^2 + \left(\bar{\nabla}_y\phi(t,\bar{y})\right)^2 + m^2\phi(t,\bar{y})^2\right)\right]$$

$$\frac{1}{2}\int d^3y \left[\Pi(t,\bar{x}), \left(\bar{\nabla}_y\phi(t,\bar{y})\right)^2\right] = \frac{1}{2}\int d^3y \left\{ \left[\Pi(t,\bar{x}), \bar{\nabla}_y\phi(t,\bar{y})\right]\cdot\bar{\nabla}_y\phi(t,\bar{y}) + \bar{\nabla}_y\phi(t,\bar{y})\left[\Pi(t,\bar{x}), \bar{\nabla}_y\phi(t,\bar{y})\right]\right\} =$$

$$= \frac{1}{2}\int d^3y \left\{ \bar{\nabla}_y\Big(\underbrace{\left[\Pi(t,\bar{x}), \phi(t,\bar{y})\right]}_{-i\delta^{(3)}(\bar{x}-\bar{y})}\Big)\cdot\bar{\nabla}_y\phi(t,\bar{y}) + \bar{\nabla}_y\phi(t,\bar{y})\cdot\bar{\nabla}_y\Big(\underbrace{\left[\Pi(t,\bar{x}), \phi(t,\bar{y})\right]}_{-i\delta^{(3)}(\bar{x}-\bar{y})}\Big)\right\} =$$

$$= +\frac{i}{2}\int d^3y\, 2\,\delta^{(3)}(\bar{x}-\bar{y})\,\bar{\nabla}_y^2\phi(t,\bar{y}) = i\,\bar{\nabla}^2\phi(t,\bar{x})$$

$$\frac{1}{2}\int d^3y \left[\Pi(t,\bar{x}), \phi^2(t,\bar{y})\right] = \frac{1}{2}\int d^3y \left\{ \left[\Pi(t,\bar{x}), \phi(t,\bar{y})\right]\phi(t,\bar{y}) + \phi(t,\bar{y})\left[\Pi(t,\bar{x}), \phi(t,\bar{y})\right]\right\} =$$

$$= -\frac{i}{2}\cdot 2\,\phi(t,\bar{x}) = -i\,\phi(t,\bar{x})$$

$$\Downarrow$$

$$\underbrace{i\left[\Pi(t,\bar{x}), H\right] = \left(-\bar{\nabla}^2 + m^2\right)\phi(t,\bar{x})}$$

$$\partial_t \;\Big|\quad \dot{\phi}(t,\bar{x}) = \Pi(t,\bar{x}) = -i\left[\phi(t,\bar{x}), H\right]$$

$$\ddot{\phi}(t,\bar{x}) = \dot{\Pi}(t,\bar{x}) = -i\left[\dot{\phi}(t,\bar{x}), H\right] = -i\left[\Pi(t,\bar{x}), H\right] = \left(\bar{\nabla}^2 - m^2\right)\phi(t,\bar{x}) \qquad (\partial_t H = 0)$$

$$\ddot{\phi}(t,\bar{x}) = \dot{\Pi}(t,\bar{x}) = -i\,[\dot{\phi}(t,\bar{x}), H] = -i\,[\Pi(t,\bar{x}), H] \overset{\vee}{=} -(-\bar{\nabla}^2 + m^2)\,\phi(t,\bar{x}) \qquad (\partial_t H = 0)$$

$$\Downarrow$$

$$(\partial_t^2 - \bar{\nabla}^2 + m^2)\,\phi(t,\bar{x}) = 0$$

$$(\Box + m^2)\,\phi(t,\bar{x}) = 0$$

$$\boxed{\dot{\Pi}(t,\bar{x}) = -i\,[\Pi(t,\bar{x}), H]}$$

observables : $\langle phys \,|\, O \,|\, phys' \rangle$ $\Rightarrow$ "representations", "pictures"

- Schrödinger picture : operators are constant, states evolve in time
- Heisenberg picture : state vectors are constant, operators evolve

└ more suitable for quantum field theory

$$i\,\partial_t O^H(t) = [O^H, H] \qquad \text{— Heisenberg equation}$$

└ time independent Hamilton operator (closed system), $H = H^+$

$$\downarrow$$

$$O^H(t) = e^{iHt}\, O^H(0)\, e^{-iHt}$$

$$\left\{ i\,\partial_t O^H(t) = i\,iH \underbrace{e^{iHt}\, O^H(0)\, e^{-iHt}}_{O^H(t)} + i \underbrace{e^{iHt}\, O^H(0)\,(-i)H\, e^{-iHt}}_{-i\,O^H(t)\,\cdot\, H} = (O^H H - H O^H) = [O^H(t), H] \right\}$$

$$\{ t_0 = 0 \rightarrow t_0 \neq 0 \}$$

state vectors : $|\alpha,t\rangle^H = |\alpha,0\rangle^H \equiv |\alpha\rangle^H$

Schrödinger picture

$$O^S = e^{-iHt} O^H e^{iHt} \quad \text{and} \quad |\alpha,t\rangle^S = e^{-iHt} |\alpha,t\rangle^H = e^{-iHt} |\alpha\rangle^H$$

Both pictures agree at $t=0$:

- O^S is time independent $\quad O^S = O^H(0)$

- $|\alpha,0\rangle^S = |\alpha\rangle$

The state vectors satisfy the Schrödinger equation

$$i\partial_t |\alpha,t\rangle^S = H|\alpha,t\rangle^S$$

$$\left\{ i\partial_t e^{-iHt}|\alpha\rangle = i(-i)He^{-iHt}|\alpha\rangle = H|\alpha,t\rangle^S \right\}$$

- The unitary transformation $O^H \to \underbrace{e^{-iHt} O^H e^{+iHt}}_{O^S}$, $|\alpha\rangle^H \to \underbrace{e^{-iHt}|\alpha\rangle^H}_{|\alpha,t\rangle^S}$

leaves matrix elements invariant:

$$^H\langle\alpha| O^H(t)|\beta\rangle^H \to {}^S\langle\alpha,t| O^S |\beta,t\rangle^S = \langle\alpha| e^{iHt} \cdot e^{-iHt} O^H e^{iHt} e^{-iHt} |\beta\rangle^H = {}^H\langle\alpha| O^H(t) |\beta\rangle^H$$

- $H^H = H^S \quad \left\{ H^S = e^{-iHt} H^H e^{iHt} = H^H \right\}$

- $O^H \to e^{-iHt} O^H e^{iHt}$, $|\alpha\rangle^H \to e^{-iHt}|\alpha\rangle^H$ is a canonical transformation, so that it leaves the commutation relations invariant

$$e^{-iHt} \mid [A^H, B^H] = C^H \mid e^{iHt}$$

$$e^{-iHt} A^H \underset{\uparrow}{} B^H e - e^{-iHt} B^H A^H e^{iHt} = C^S \Rightarrow [A^S, B^S] = C^S$$

$$e^{iHt} e^{-iHt}$$

Manifest Lorentz invariance is easier to kept in the Heisenberg picture

Interaction picture (Dirac, Tomonaga)

interactions

assume $$H = H_0 + H_1$$

free Hamiltonian

$$O^I(t) \equiv e^{iH_0^S t} O^S e^{-iH_0^S t}$$

$$|\alpha, t\rangle^I \equiv e^{iH_0^S t} |\alpha, t\rangle^S$$

$$O^I(t) = e^{iH_0 t} e^{-iHt} O^H(t) e^{iHt} e^{-iH_0^S t}$$

$$|\alpha, t\rangle^I = e^{iH_0^S t} e^{-iHt} |\alpha\rangle^H$$

{ if $H_1 = 0$, then the interaction picture coincides with the Heisenberg one }

The Schrödinger equation in the interaction picture

$$i\partial_t |\alpha, t\rangle^I = i i H_0^S e^{iH_0^S t} |\alpha, t\rangle^S + i e^{iH_0^S t} (-i) \underbrace{i\partial_t |\alpha, t\rangle^S}_{= H|\alpha, t\rangle^S} = i^2 \underbrace{H_0^S}_{= H_0^I} |\alpha, t\rangle^I - i^2 \underbrace{e^{iH_0^S t} H e^{-iH_0^S t}}_{= H^I} \underbrace{e^{iH_0^S t} |\alpha, t\rangle^S}_{= |\alpha, t\rangle^I} =$$

$$= (H^I - H_0^I)|\alpha, t\rangle^I = H_1^I |\alpha, t\rangle^I$$

$$= (H' - H_0')|\alpha, t\rangle' = H_1'\,|\alpha, t\rangle'$$

$$\boxed{i\partial_t |\alpha, t\rangle' = H_1'\,|\alpha, t\rangle'}$$

state vector evolve according to the interaction Hamiltonian

$$i\partial_t O' = i\partial_t\left[e^{iH_0^s t} O^s e^{-iH_0^s t}\right] = i\Big[iH_0^s \underbrace{e^{iH_0^s t} O^s e^{-iH_0^s t}}_{O'} - i\underbrace{e^{iH_0^s t} O^s H_0^s e^{-iH_0^s t}}_{O' \cdot H_0^s}\Big] = [O', H_0^s]$$

$$\boxed{i\partial_t O' = [O', H_0^s]}$$

operator evolve according to the free Hamiltonian

The time evolution operator $U(t_1, t_0)$:

$$|\alpha, t_1\rangle' = U(t_1, t_0)\,|\alpha, t_0\rangle'$$

A formal solution:

$$|\alpha, t_1\rangle' = e^{iH_0^s t_1} \underbrace{|\alpha, t_1\rangle^s}_{e^{-iH(t_1 - t_0)}|\alpha, t_0\rangle^s} = \underbrace{e^{iH_0^s t_1} e^{-iH(t_1 - t_0)} e^{-iH_0^s t_0}}_{U(t_1, t_0)} |\alpha, t_0\rangle'$$

$$H_0^s = H_0'$$

$$\boxed{U(t_1, t_0) = e^{iH_0 t_1} e^{-iH(t_1 - t_0)} e^{-iH_0 t_0}}$$

Properties:

- $U(t_0, t_0) = \mathbb{1}$
- $U(t_2, t_1)\, U(t_1, t_0) = e^{iH_0 t_2} e^{-iH(t_2 - t_1)} \underbrace{e^{-iH_0 t_1} e^{iH_0 t_1}}_{\mathbb{1}} e^{-iH(t_1 - t_0)} e^{-iH_0 t_0} = e^{iH_0 t_2} e^{-iH(t_2 - t_0)} e^{-iH_0 t_0} =$

$$= U(t_2, t_0)$$

- for $t_2 = t_0$ one finds $\quad U(t_0, t_1)\, U(t_1, t_0) = \mathbb{1}$

$$\Downarrow$$

$$U(t_1, t_0)^{-1} = U(t_0, t_1)$$

- $U^{\dagger}(t_1, t_0) = e^{+iH_0^{\dagger} t_0} e^{+iH^{\dagger}(t_1 - t_0)} e^{-iH_0^{\dagger} t_1} = U(t_0, t_1) = U^{-1}(t_1, t_0)$ (unitarity)

$U(t, t_0)$ satisfies the following differential equation:

$$\boxed{i\partial_t U(t, t_0) = H_1'(t)\, U(t, t_0) \quad \text{with} \quad U(t_0, t_0) = \mathbb{1}}$$

Proof

$i\partial_t U(t, t_0) = i\partial_t \left[e^{iH_0 t} e^{-iH(t - t_0)} e^{-iH_0 t_0} \right] = i \left[iH_0 e^{iH_0 t} e^{-iH(t - t_0)} + e^{iH_0 t} (\underset{H_0 + H_1}{\underset{\parallel}{-iH}}) e^{-iH(t - t_0)} \right] e^{-iH_0 t_0} =$

$$= -H_0 \cancel{U(t, t_0)} + H_0 \cancel{U(t, t_0)} + e^{iH_0 t} H_1 e^{-iH(t - t_0)} e^{-iH_0 t_0} =$$

$$= e^{iH_0 t} H_1 e^{-iH_0 t} \underbrace{e^{iH_0 t} e^{-iH(t - t_0)} e^{-iH_0 t_0}}_{U(t, t_0)} = H_1'(t)\, U(t, t_0)$$

$$\int^{t} \Big| \quad i\partial_{t'} U(t', t_0) = H_1'(t')\, U(t', t_0)$$

$$\int_{t_0} \Big| \quad i\partial_{t'} U(t',t_0) = H_1'(t')\, U(t',t_0)$$

$U(t,t_0)$

$$i\, U(t,t_0) - i\mathbb{1} = \int_{t_0}^{t} dt'\, H_1'(t')\, U(t',t_0)$$

$$\boxed{U(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt'\, H_1'(t')\, U(t',t_0)}$$

By iteration:

$U^{(0)}(t,t_0) = \mathbb{1} \longrightarrow$ to the rhs

$U^{(1)}(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, H_1'(t_1) \longrightarrow$ to the rhs

$$U^{(2)}(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, H_1'(t_1)\left[\mathbb{1} - i\int_{t_0}^{t_1} dt_2\, H_1'(t_2)\right] = \mathbb{1} + (-i)\int_{t_0}^{t} dt_1\, H_1'(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1'(t_1)\, H_1'(t_2)$$

$\vdots$

$$U(t,t_0) = \mathbb{1} + \cdots + (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_1'(t_1)\cdot H_1'(t_2)\cdot \ldots \cdot H_1'(t_n) + \cdots$$

Time ordering product

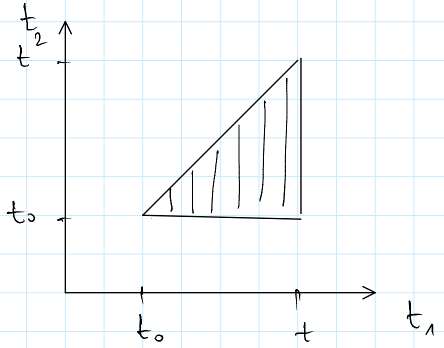
$$T\{H_1(t_1)\cdot H_1(t_2)\cdot \ldots \cdot H_1(t_n)\} = H_1(t_{i_1})\, H(t_{i_2}) \cdots H(t_{i_n}) \quad \text{such that}$$

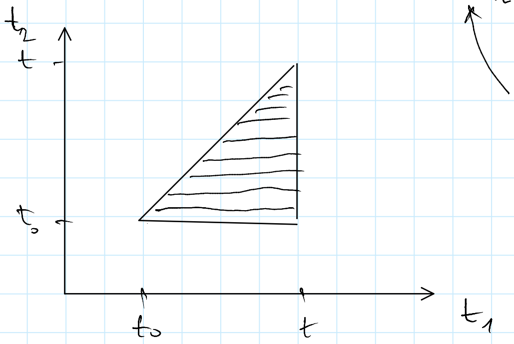
$$t_{i_1} \geq t_{i_2} \geq \ldots \geq t_{i_n}$$

Let's consider $U^{(2)}(t,t_0)$

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1(t_1)\, H_1(t_2) = \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1\, H_1(t_1)\, H(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, H(t_2)\, H(t_1)$$

$t_1 \Leftrightarrow t_2$





We add the above two equivalent expressions:

$$2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1(t_1)\, H(t_2) = \underbrace{\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_1(t_1)\, H_1(t_2)}_{t_2 \leq t_1} + \underbrace{\int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, H_1(t_2)\, H_1(t_1)}_{t_2 \geq t_1} =$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \underbrace{T\{ H_1(t_1)\, H_1(t_2)\}}$$

$$\hookrightarrow H_1(t_1)\, H(t_2)\, \theta(t_1 - t_2) + H_1(t_2)\, H_1(t_1)\, \theta(t_2 - t_1)$$

The same for the higher order terms ($n!$ possible permutations)

$$n! \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n\, H_1(t_1) \cdots H_1(t_n) = \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n\, T\{H_1(t_1) \cdots H_1(t_n)\}$$

The perturbation series for the evolution operator :

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\{ H_1(t_1) \cdot \ldots \cdot H_1(t_n) \}$$

- Show that the above series indeed satisfies the equation for $U(t,t_0)$.

$$i\partial_t U(t,t_0) = H_1(t)\, U(t,t_0)$$

$$i\partial_t U(t,t_0) = i \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_{n-1} \, T\{ H_1(t_1) \cdot \ldots \cdot H_1(t_{n-1}) H_1(t) \} =$$

note that $t \geq t_i$

$$= H_1(t) \sum_{n=1}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_{n-1} \, T\{ H_1(t_1) \cdot \ldots \cdot H_1(t_{n-1}) \} =$$

$$i(-i)^n = i(-i)(-i)^{n-1} = (-i)^{n-1} \quad , \qquad n-1 = n'$$

$$= H_1(t)\, U(t,t_0)$$

Formally we can write $\quad U(t,t_0) = T\left\{ e^{-i\int_{t_0}^{t} dt' H_1(t')} \right\} = T\left\{ e^{-i\int_{t_0}^{t} d^4x' \mathcal{H}(x')} \right\}$

The scattering matrix

- Let $|\psi(t)\rangle$ denot the time-dependent state vector, that has evolved from free initial state $|\phi_i\rangle$ at $t \to -\infty$

$$\lim_{t\to-\infty} |\Psi(t)\rangle = |\phi_i\rangle$$

- the S-matrix:

$$S_{fi} = \lim_{t\to+\infty} \langle \phi_f | \Psi(t)\rangle \equiv \langle \phi_f | S | \phi_i \rangle$$

definition of the S operator

$$\Downarrow$$

$$S_{fi} = \lim_{t_2\to+\infty} \lim_{t_1\to-\infty} \langle \phi_f | U(t_2, t_1) | \phi_i \rangle$$

$$S = U(+\infty, -\infty)$$

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n \, T\{ H_1(t_1) \cdots H_1(t_n) \} \quad , \quad SS^+ = S^+ S = 1\!\!1$$

The Wick's theorem

$2 \to 2$ scattering : $k_1, k_2 \to k_3 k_4$

$$S_{fi} = \langle f | S | i \rangle \propto \langle \phi | a_{k_4} a_{k_3} S a^+_{k_1} a^+_{k_2} | \phi \rangle$$

To find S_{fi} one has to select contributions to S of the form

$$f(l_1, l_2, l_3, l_4) \; a^+_{l_4} a^+_{l_3} a_{l_1} a_{l_2},$$

it could be done manually, it is tedious, the Wick's theorem is to help

- we have to calculate $\langle phys | T\{ \phi_{A_1}(x_1) \cdots \phi_{A_k}(x_k) \cdots \} | phys \rangle$

1° $T\{ \phi_A(x_1)\, \phi_B(x_2) \} = :\phi_A(x_1)\, \phi_B(x_2): + \underbrace{\langle 0 | T\{ \phi_A(x_1)\, \phi_B(x_2) \} | 0 \rangle}$

$\equiv \phi_A(x_1)\, \phi_B(x_2)$ (the contraction)

2° $T\{ \phi_A(x_1)\, \phi_B(x_2)\, \phi_C(x_3) \} = ?$

assume $t_1, t_2 > t_3$ then

$$T\{ \phi_A(x_1)\, \phi_B(x_2) \}\, \phi_C(x_3) = \underbrace{:\phi_A(x_1)\, \phi_B(x_2):\, \phi_C(x_3)} + \phi_A(x_1)\, \phi_B(x_2)\, \phi_C(x_3)$$

$$\left(\phi_A^- \phi_B^- + \phi_A^- \phi_B^+ + \phi_A^+ \phi_B^+ + \phi_B^- \phi_A^+ \right) \cdot \left(\phi_C^- + \phi_C^+ \right)$$

$$\phi_A^- \phi_B^+ \phi_C^- = \phi_A^- \phi_C^- \phi_B^+ + \phi_A^- [\phi_B^+, \phi_C^-]$$

$$\phi_A^+ \phi_B^+ \phi_C^- = \phi_A^+ \left(\phi_C^- \phi_B^+ + [\phi_B^+, \phi_C^-] \right) = \phi_C^- \phi_A^+ \phi_B^+ + [\phi_A^+, \phi_C^-]\, \phi_B^+ + \phi_A^+ [\phi_B^+, \phi_C^-] =$$

$$\phi_B^- \phi_A^+ \phi_C^- = \phi_B^- \phi_C^- \phi_A^+ + \phi_B^- [\phi_A^+, \phi_C^-]$$

So we got

$$:\phi_A \phi_B: \phi_C = :\phi_A \phi_B \phi_C: + \underbrace{(\phi_A^- + \phi_A^+)}_{\phi_A} [\phi_B^+, \phi_C^-] + \underbrace{(\phi_B^- + \phi_B^+)}_{\phi_B} [\phi_A^+, \phi_C^-]$$

$$D(x-y) = [\phi^+(y), \phi^-(x)]\, \Theta(y^0 - x^0) + \underbrace{[\phi^+(x), \phi^-(y)]}_{\text{c-number}}\, \Theta(x^0 - y^0)$$

For $y^0 > x^0$ $\quad D(x-y) = [\phi^+(y), \phi^-(x)]$

$$\langle \phi |\quad T\{\phi(x)\phi(y)\} = :\phi(x)\phi(y): + D(x-y) \quad |\phi\rangle$$

$$\langle \phi | T\{\phi(x)\phi(y)\} | \phi\rangle = D(x-y) \langle \phi | \phi\rangle = D(x-y)$$

$$\Downarrow$$

$$[\phi^+(y), \phi^-(x)] = \langle \phi | T\{\phi(x)\phi(y)\} | \phi\rangle$$

$$:\phi_A \phi_B \phi_C: + \phi_A \langle 0 | T\{\phi_B \phi_C\} | 0\rangle + \phi_B \langle 0 | T\{\phi_A, \phi_C\} | 0\rangle$$

$$T\{\phi_A(x_1)\phi_B(x_2)\phi_C(x_3)\} = :\phi_A(x_1)\phi_B(x_2)\phi_C(x_3): + :\underbrace{\phi_A(x_1)\phi_B(x_2)}\phi_C(x_2): +$$

$$+ :\phi_A(x_1)\underbrace{\phi_B(x_2)\phi_C(x_2)}: + :\underbrace{\phi_A(x_1)\phi_B(x_2)}\phi_C(x_3):$$

The Wick's theorem

A time-ordered product of operators can be decomposed into a sum of corresponding contracted normal products. All possible contractions appear in the sum

$$T\{ABC \cdots XYZ\} = :ABC \cdots XYZ: +$$

$$+ :ABC \cdots XYZ: + :ABC \cdots XYZ: + \cdots + :ABC \cdots XYZ: +$$

$$+ :ABCD \cdots XYZ: + :ABCDE \cdots XYZ: + \cdots :ABC \cdots WXYZ: + :ABCD \cdots XYZ: +$$

$$+ \text{ triple contractions } + \cdots$$

For fermionic fields one has to modify the definition of contraction:

$$:ABCDEF \cdots KLM \cdots: = \epsilon_P :ABF \cdots KM \cdots: CE \; DL \,,$$

where $\epsilon_P = \pm 1$ is the corresponding sign of necessary permutation

Remarks:

- except for a possible sign factor the order of operators within the argument of a normal order does not matter.

- except for a possible sign factor the order of operators within the argument of a normal order does not matter:

$$:\hat{\phi}_A \hat{\phi}_B: = \epsilon_{AB} :\hat{\phi}_B \hat{\phi}_A: \quad \text{where } \epsilon_{AB} = \begin{cases} 1 & \text{for } \hat{\phi}_A, \hat{\phi}_B \text{ bosonic} \\ -1 & \hat{\phi}_A, \hat{\phi}_B \text{ fermionic} \end{cases}$$

- time-ordered product is also invariant under permutations, except for a sign factor:

$$T\{\hat{\phi}_A(x)\, \hat{\phi}_B(y)\} = \epsilon_{AB}\, T\{\hat{\phi}_B(y)\, \hat{\phi}_A(x)\}$$

Propagators

scalar field

$$D_F(x-y) = \langle 0 | T\{\phi(x)\, \phi^+(y)\} | 0 \rangle = \underbrace{\phi(x)\, \phi^+}(y)$$

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}}$$

Dirac field

$$S_F(x-y) = \langle 0 | T\{\psi(x)\, \bar{\psi}(y)\} | 0 \rangle =,$$

4x4 matrix in the Dirac indices

$$= \underbrace{\psi(x)\, \bar{\psi}}(y)$$

$$\langle 0 | T\{\psi_\alpha(x)\, \bar{\psi}_\beta(y)\} | 0 \rangle$$

$$S_F(x-y) = \int \frac{d^4p}{\ } \frac{i}{\ } e^{-ip(x-y)}$$

$$\boxed{S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{\not p - m + i\epsilon}} e^{-ip(x-y)}}$$

$i\,(\not p - m + i\epsilon)^{-1}$ inverse matrix

$$i\,\frac{\not p + m}{p^2 - m^2 + i\epsilon} = \frac{i}{\not p - m + i\epsilon}\,\frac{\not p + m}{\not p + m}$$

$$(\not p - m)(\not p + m) = p^\mu p^\nu \gamma_\mu \gamma_\nu - m^2 = \frac{1}{2} p^\mu p^\nu \{\gamma_\mu, \gamma_\nu\} - m^2 =$$

$$= p^\mu p^\nu \eta_{\mu\nu} - m^2 = p^2 - m^2$$

– Show that $(i\not\partial_x - m)\, S_F(x-y) = i\,\delta^{(4)}(x-y)$

$$(i\not\partial_x - m) \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not p - m + i\epsilon} e^{-ip(x-y)} = i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left\{ i\gamma^\mu \frac{1}{\not p - m + i\epsilon} (-i) p_\mu - \frac{m}{\not p - m + i\epsilon} \right\} =$$

$$= i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{\not p - m}{\not p - m + i\epsilon} = i\,\delta^{(4)}(x-y)$$

U(1) gauge field

$$D_F^{\mu\nu}(x-y) = \langle 0 | T\{A^\mu(x) A^\nu(y)\} | 0 \rangle = \underbrace{A^\mu(x) A^\nu(y)}$$

$$\boxed{D_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2+i\epsilon} e^{-ip(x-y)} \eta^{\mu\nu}}$$